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LETTER TO THE EDITOR

Exact solitary waves in a convecting fluid

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Abstract. The perturbed Korteweg–de Vries equation $u_t+\lambda_1 uu_x+\lambda_2 u_{xxx}+\lambda_3 u_{xxxx}+\lambda_4(uu_x)_x+\lambda_5 u_{xx}=0$, which describes the evolution of long shallow waves in a convecting fluid when the critical Rayleigh number slightly exceeds its critical value, admits two types of exact solitary wave solution.

The following perturbed Korteweg–de Vries equation

$$u_t+\lambda_1 u_{xx}+\lambda_2 u_{xxx}+\lambda_3 u_{xxxx}+\lambda_4(uu_x)_x+\lambda_5 u_{xx}=0 \tag{1}$$

can be used to describe the nonlinear behaviour of a convecting fluid near the transition point [1], where the parameters λ_i

$$\lambda_1=\frac{3}{2\sigma G}(10+\sigma G) \qquad \lambda_2=\frac{\sigma\sqrt{G}}{2}\left(\frac{1}{3}+\frac{34}{21}\sigma\right)$$
$$\lambda_3=\varepsilon\sigma\frac{682\sigma^2G+717}{2079} \qquad \lambda_4=\frac{8\varepsilon}{\sqrt{G}} \qquad \lambda_5=\frac{\sigma R_2}{15}\varepsilon \tag{2}$$

where σ is the Prandtl number, G the Galileo number, and ε a small parameter such that the excess of the Rayleigh number above its critical value is given by $\varepsilon^2 R_2$.

It is interesting that various important equations such as KdV, Burgers, KdV-Burgers, Kuramoto–Sivashinsky [2–5] equations are just special cases of (1). Aspe and Depassier [1] have studied the asymptotic solitary wave solution of (1). In this letter we give some exact solitary wave solutions in a simple way. The result shows that the exact solutions are completely different from the perturbative solutions.

For the travelling wave solutions $u(x,t)=u(\xi)=u(kx+\omega t)$, equation (1) becomes

$$\omega u+\tfrac{1}{2}k\lambda_1 u^2+\lambda_2 k^3 u_{\xi\xi}+k^2\lambda_5 u_\xi+\lambda_3 k^4 u_{\xi\xi\xi}+\tfrac{1}{2}\lambda_4 k^2(u^2)_\xi+C=0 \tag{3}$$

after integrating once with respect to ξ, where C is the integral constant.

Noticing that

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\sum_{j=0}^{n}A_j\tanh^j\xi=\sum_{j=0}^{n+1}B_j\tanh^j\xi \tag{4}$$

$$\left(\sum_{j=0}^{n}A_j\tanh^j\xi\right)^2=\sum_{j=0}^{2n}C_j\tanh^j\xi \tag{5}$$

where A_j ($j=0, 1, \ldots, n$), B_j ($j=0, 1, \ldots, n+1$) and C_j ($j=0, 1, \ldots, 2n$) are constants, we make the crucial ansatz for $\lambda_4 \neq 0$:

$$u(\xi) = \sum_{j=0}^{2} A_j \tanh^j \xi. \tag{6}$$

By substituting (5) into (2) and equating the coefficients of $\tanh^j \xi$ ($j=0, 1, \ldots, 5$), we obtain a set of equations

$$\lambda_4 A_2 + 12\lambda_3 k^2 = 0 \tag{7}$$

$$\tfrac{1}{2}k\lambda_1 A_2^2 + 6\lambda_2 k^3 A_2 - 6A_1\lambda_3 k^4 - 3A_1 A_2 k^2 \lambda_4 = 0 \tag{8}$$

$$k\lambda_1 A_1 A_2 + 2A_1\lambda_2 k^3 - 2A_2 k^2 \lambda_5 + 40A_2\lambda_3 k^4 + \lambda_4 k^2(2A_2^2 - A_1^2 - 2A_0A_2) = 0 \tag{9}$$

$$\omega A_2 + \tfrac{1}{2}k_1\lambda_1(A_1^2 + 2A_0A_2) - 8A_2\lambda_2 k^3 - A_1 k^2 \lambda_5 + 8A_1\lambda_3 k^4 + (3A_1A_2 - A_0A_1)\lambda_4 k^2 = 0 \tag{10}$$

$$\omega A_1 + k\lambda_1 A_1 A_0 - 2A_1\lambda_2 k^3 + 2A_2\lambda_5 k^2 - 16A_2\lambda_3 k^4 + \lambda_4 k^2(2A_0A_2 + A_1^2) = 0 \tag{11}$$

$$C = -\omega A_0 - \tfrac{1}{2}k\lambda_1 A_0^2 - 2A_2\lambda_2 k^3 - A_1 k^2 \lambda_5 + 2A_1\lambda_3 k^4 - \lambda_4 k^2 A_0 A_1 = 0. \tag{12}$$

There exist two cases to solve equations (7)-(12):

(i) $\lambda_1\lambda_3 \neq \lambda_2\lambda_4$. In this case, the solutions of (7)-(12) are

$$k = \pm\frac{\lambda_2\lambda_4 - \lambda_1\lambda_3}{10\lambda_3\lambda_4} \tag{14}$$

$$A_0 = -\frac{\lambda_5}{\lambda_4} + \frac{1}{25\lambda_3\lambda_4^3}(\lambda_2\lambda_4 - \lambda_1\lambda_3)(26\lambda_1\lambda_3 - \lambda_2\lambda_4) \tag{15}$$

$$A_1 = \pm\frac{6(\lambda_2\lambda_4 - \lambda_1\lambda_3)^2}{25\lambda_3\lambda_4^3} \tag{16}$$

$$A_2 = -\frac{6(\lambda_2\lambda_4 - \lambda_1\lambda_3)^2}{5\lambda_3\lambda_4^3} \tag{17}$$

$$\omega = \pm\frac{\lambda_1(\lambda_2\lambda_4 - \lambda_1\lambda_3)}{40\lambda_3\lambda_4^4}\left(4\lambda_5 + \frac{1}{5\lambda_3}(\lambda_2\lambda_4 - \lambda_1\lambda_3)(2\lambda_2\lambda_4 - 15\lambda_1\lambda_3)\right) \tag{18}$$

and the integral constant C is given by (12). It is easy to prove that the solitary wave solution (6) with (14)-(18) has a monotonic kink-shaped form. The amplitude $2A_1$, the wave number k and the velocity ω/k of this kink-like solution are completely fixed by the model parameters. It is clear that the solitary wave (6) is ε dependent in the following way:

$$A_0 \sim \varepsilon^{-2} \qquad A_1 \sim \varepsilon^{-2} \qquad A_2 \sim \varepsilon^{-2} \qquad k \sim \varepsilon^{-1} \qquad \omega \sim \varepsilon^{-3} \tag{19}$$

after returning to the original model parameters for a convecting fluid given in (2). Equation (19) tells us that the solitary wave solution obtained here is non-perturbative.

(ii) $\lambda_1\lambda_3=\lambda_2\lambda_4$. In this case, the solutions of (7)-(11) are

$$A_0=\frac{1}{\lambda_4}[-\lambda_5+8k^2\lambda_3] \tag{21}$$

$$A_1=0 \tag{22}$$

$$A_2=-\frac{12}{\lambda_4}\lambda_3k^2=-\frac{12}{\lambda_1}\lambda_2k^2 \tag{23}$$

$$\omega=\frac{\lambda_2\lambda_5}{\lambda_3} \tag{24}$$

where C is still given by (12) but with A_0, A_A, A_2 and ω being given by (21)-(24), and k remains as an arbitrary constant. It is worthwhile to mention that in this case the dispersion relation (24) is only a linear one!

Actually, in this case, for the travelling wave solution (1) with the linear dispersion relation (24) can be rewritten as

$$\left(\lambda_1+\frac{\lambda_3\lambda_4}{\lambda_2\lambda_5}\omega\frac{\partial}{\partial\xi}\right)\left(\frac{\lambda_2\lambda_5}{\lambda_3}u_\xi+\lambda_1uu_\xi+\lambda_2k^2u_{\xi\xi\xi}\right)=0. \tag{25}$$

The general solution of (20) with boundary condition $u(\pm\infty)<\infty$ can be expressed by

$$\xi+\xi_0=\pm\int^u\frac{(-2/\lambda_2k^2)^{1/2}\,\mathrm{d}u}{(\lambda_1u^3/3+\lambda_2\lambda_5u^2/2\lambda_3+Cu+D)^{1/2}} \tag{26}$$

where k, C, D and ξ_0 are arbitrary constants. The solitary wave solution (6) with (21)-(24) is just a special case of (26). Though this type of the travelling wave solution possesses some arbitrary constants, the velocity of the solution is constant because of (24). In this case, the travelling wave solution (26) or (6) with (21)-(24) is ε independent and so it is also non-perturbative.

All the possible travelling wave solutions for the $\lambda_4=0$ case can be obtained by modifying the crucial ansatz (6) as

$$u(\xi)=\sum_{j=0}^{3}A_j\tanh^j\xi \tag{27}$$

but we do not discuss them further here because various authors have obtained solitary wave solutions of this form by other approaches. For instance, Kudryashov [6] has obtained this type of solutions for $\lambda_4=0$ by the Weiss-Tabor-Carnevale method [7].

In summary, the perturbed KdV equation (1) which can be used to describe the evolution of long shallow waves when the Rayleigh number slightly exceeds its critical value allows two types of solitary wave solutions. The first type of solution is the kink-shaped solitary wave solution, and this type of solution is valid only for $\lambda_1\lambda_3\neq\lambda_2\lambda_4$. When $\lambda_1\lambda_3=\lambda_2\lambda_4$, the model admits another type of solitary wave solution which possesses an ε-independent bell-shaped form. The dispersion relation in this type of travelling wave solution is only a linear one. These two types of solutions are all non-perturbative and the velocities of them are all fixed by the model parameters λ_i $(i=1,\ldots,5)$.

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